On the Noether Invariance Principle for Constrained Optimal Control Problems*

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Abstract

We obtain a generalization of Noether's invariance principle for optimal control problems with equality and inequality state-input constraints. The result relates the invariance properties of the problems with the existence of conserved quantities along the constrained Pontryagin extremals. A result of this kind was posed as an open question by Vladimir Tikhomirov, in 1986.

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1 Introduction

Noether's invariance principle is one of the most helpful and fundamental results of physics. It describes the universal fact that "invariance with respect to some family of parameter transformations gives rise to the existence of certain conserved quantities." Such relation is used to explain everything from the fusion of hydrogen to the motion of planets orbiting the sun [6]. For a modern account of Noether's invariance principle, in the context of the calculus of variations, we refer the reader to [1, 10]. Extensions for the unconstrained problems of optimal control are available in [8, 9]. Here we generalize the previous results

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[8, 9] to cover both holonomic and nonholonomic constraints. The motivation for the present study was Tikhomirov's book [7, Sec. 4.3]: *Presumably, it can be shown for a sufficiently broad class of extremal problems involving constraints that Noether's invariance theorem is still valid. But general results of this kind have, to date, not been obtained.* Theorem 3.1 provides such general result.

2 Constrained Optimal Control Problems and Optimality Conditions

We deal with a broad class of extremal problems in the calculus of variations and optimal control involving equality and/or inequality constraints. We consider a nonlinear control system,

$$\dot{x}(t) = \varphi\left(t, x(t), u(t)\right) , \tag{1}$$

of n differential equations, subject to $m - m'$ equality constraints,

$$\phi_i\left(t, x(t), u(t)\right) = 0 \, , \quad i = 1, \ldots, m - m' \, , \tag{2}$$

m' inequality constraints,

$$\phi_j\left(t, x(t), u(t)\right) \geq 0 \, , \quad j = m - m' + 1, \ldots, m \, , \tag{3}$$

and $2n$ boundary conditions

$$x(a) = \alpha \, , \quad x(b) = \beta \, . \tag{4}$$

The problem is to find a piecewise-continuous control function $u(\cdot) = (u_1(\cdot), \ldots, u_r(\cdot))$, and the corresponding state trajectory $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))$, satisfying (1), (2), (3), and (4), which minimizes or maximizes the integral cost functional

$$I[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) \mathrm{d}t \, .$$

This problem is denoted in the sequel by (P). Both the initial time a and terminal time b, $a < b$, are fixed. The boundary values α, $\beta \in \mathbb{R}^n$ are also given. The functions $L(\cdot, \cdot, \cdot)$, $\varphi(\cdot, \cdot, \cdot)$ and $\phi(\cdot, \cdot, \cdot)$ are assumed to be continuously differentiable with respect to all variables. We shall also assume that the Jacobian of the constraints (2) and (3), $\frac{\partial \phi}{\partial u}(t, x, u)$, has full rank for all $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^r$ [4, Ch. 6]. Many practical applications of problem (P) appear in engineering and economics. We refer the interested reader to [2] and references therein.

The celebrated Pontryagin's maximum principle [4] gives necessary optimality conditions to be satisfied by the solutions of optimal control problems.

Theorem 2.1 (Pontryagin Maximum Principle for (P)). *Let $u(t)$, $t \in [a, b]$, be an optimal control for the constrained optimal control problem (P), and $x(\cdot)$ the corresponding state trajectory. Then there exists a constant $\psi_0 \leq 0$, a*

continuous costate n-vector function $\psi(\cdot)$ having piecewise-continuous derivatives, and (assuming that the rank condition is satisfied) piecewise-continuous multipliers $\lambda(\cdot)$, $\lambda(t) \geq 0$, satisfying the equations:

$$\begin{aligned}
\dot{x}(t) &= \frac{\partial H}{\partial \psi}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda(t)\right) , \\
\dot{\psi}(t) &= -\frac{\partial H}{\partial x}\left(t, x(t), u(t), \psi_0, \psi(t), \lambda(t)\right) , \\
\frac{\partial H}{\partial u}&\left(t, x(t), u(t), \psi_0, \psi(t), \lambda(t)\right) = 0 ,
\end{aligned}$$

along with the constraints (2)-(3) *and boundary conditions* (4)*, where the Hamiltonian H is defined by*

$$H(t, x, u, \psi_0, \psi, \lambda) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u) + \lambda \cdot \phi(t, x, u) .$$

Moreover, $H(t, x(t), u(t), \psi_0, \psi(t), \lambda(t))$ is a continuous function of t and, on each interval of continuity of $u(\cdot)$, is differentiable and satisfies the equality

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} . \tag{5}$$

For versions of Theorem 2.1 under weaker smoothness hypotheses on the data of the problem, see [3, 11].

3 Main Result

The following result asserts that the presence of an invariant structure of the optimal control problems involving equality and inequality constraints, imply that their extremals (and solutions) also possess a certain invariance. The result is expressed, as it happens for the problems of the calculus of variations [1, 10] and for the unconstrained optimal control problems [8, 9], as an instance of Noether's universal principle. Theorem 3.1 extends [9, Theorem 5.1] to the case of constrained optimal control problems.

Theorem 3.1. *If there exists a C^2-smooth one-parameter family of maps*

$$\begin{gathered}
h^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r , \\
h^s(t, x, u) = \left(T(t, x, u, s), X(t, x, u, s), U(t, x, u, s)\right) , \\
s \in (-\varepsilon, \varepsilon) ,\ \varepsilon > 0 ,
\end{gathered}$$

with $h^0(t, x, u) = (t, x, u)$ for all $(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^r$, and satisfying

$$L\left(t, x(t), u(t)\right) = L \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) , \tag{6}$$

$$\frac{d}{dt} X\left(t, x(t), u(t), s\right) = \varphi \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) , \tag{7}$$

$$\phi\left(t, x(t), u(t)\right) = \phi \circ h^s\left(t, x(t), u(t)\right) \frac{d}{dt} T\left(t, x(t), u(t), s\right) , \tag{8}$$

then,

$$\psi(t)\cdot\frac{\partial}{\partial s}\,X\left(t,x(t),u(t),s\right)\big|_{s=0}-H(t,x(t),u(t),\psi_0,\psi(t),\lambda(t))\frac{\partial}{\partial s}\,T\left(t,x(t),u(t),s\right)\big|_{s=0}$$

is constant in $t\in[a,b]$ for any quintuple $(x(\cdot),u(\cdot),\psi_0,\psi(\cdot),\lambda(\cdot))$ satisfying the Pontryagin maximum principle (Theorem 2.1), with H the Hamiltonian associated to the problem (P): $H(t,x,u,\psi_0,\psi,\lambda)=\psi_0 L(t,x,u)+\psi\cdot\varphi(t,x,u)+\lambda\cdot\phi(t,x,u)$.

Proof. Using the fact that $h^0(t,x,u)=(t,x,u)$, from condition (6) one gets

$$\begin{aligned}0&=\frac{d}{ds}\left(L\circ h^s\left(t,x(t),u(t)\right)\frac{d}{dt}T\left(t,x(t),u(t),s\right)\right)\Bigg|_{s=0}\\&=\frac{\partial L}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial L}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial L}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+L\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}\,,\end{aligned}\tag{9}$$

while condition (7) and (8) yields

$$\frac{d}{dt}\,\frac{\partial X}{\partial s}\bigg|_{s=0}=\frac{\partial\varphi}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial\varphi}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial\varphi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\varphi\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}\,,\tag{10}$$

$$0=\frac{\partial\phi}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial\phi}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial\phi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\phi\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}\,.\tag{11}$$

Multiplying (9) by ψ_0, (10) by $\psi(t)$, and (11) by $\lambda(t)$, we can write:

$$\begin{aligned}&\psi_0\left(\frac{\partial L}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial L}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial L}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+L\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}\right)\\&+\psi(t)\cdot\left(\frac{\partial\varphi}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial\varphi}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial\varphi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\varphi\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}-\frac{d}{dt}\,\frac{\partial X}{\partial s}\bigg|_{s=0}\right)\\&+\lambda(t)\cdot\left(\frac{\partial\phi}{\partial t}\,\frac{\partial T}{\partial s}\bigg|_{s=0}+\frac{\partial\phi}{\partial x}\cdot\frac{\partial X}{\partial s}\bigg|_{s=0}+\frac{\partial\phi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\phi\frac{d}{dt}\,\frac{\partial T}{\partial s}\bigg|_{s=0}\right)=0\,.\end{aligned}\tag{12}$$

According to the Pontryagin maximum principle, the function

$$\begin{aligned}\psi_0 L\left(t,x(t),U\left(t,x(t),u(t),s\right)\right)+\psi(t)\cdot\varphi\left(t,x(t),U\left(t,x(t),u(t),s\right)\right)\\+\lambda(t)\cdot\phi\left(t,x(t),U\left(t,x(t),u(t),s\right)\right)\end{aligned}$$

attains an extremum for $s=0$. Therefore

$$\psi_0\frac{\partial L}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\psi(t)\cdot\frac{\partial\varphi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}+\lambda(t)\cdot\frac{\partial\phi}{\partial u}\cdot\frac{\partial U}{\partial s}\bigg|_{s=0}=0$$

and (12) simplifies to

$$\begin{aligned}
&\psi_0 \left(\frac{\partial L}{\partial t} \left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial x} \cdot \left.\frac{\partial X}{\partial s}\right|_{s=0} + L \frac{d}{dt} \left.\frac{\partial T}{\partial s}\right|_{s=0} \right) \\
&\quad + \psi(t) \cdot \left(\frac{\partial \varphi}{\partial t} \left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial \varphi}{\partial x} \cdot \left.\frac{\partial X}{\partial s}\right|_{s=0} + \varphi \frac{d}{dt} \left.\frac{\partial T}{\partial s}\right|_{s=0} - \frac{d}{dt} \left.\frac{\partial X}{\partial s}\right|_{s=0} \right) \\
&\qquad + \lambda(t) \cdot \left(\frac{\partial \phi}{\partial t} \left.\frac{\partial T}{\partial s}\right|_{s=0} + \frac{\partial \phi}{\partial x} \cdot \left.\frac{\partial X}{\partial s}\right|_{s=0} + \phi \frac{d}{dt} \left.\frac{\partial T}{\partial s}\right|_{s=0} \right) = 0 \, . \qquad (13)
\end{aligned}$$

From the adjoint system $\dot{\psi} = -\frac{\partial H}{\partial x}$ and the equality (5), we know that

$$\begin{gathered}
\dot{\psi} = -\psi_0 \frac{\partial L}{\partial x} - \psi \cdot \frac{\partial \varphi}{\partial x} - \lambda \cdot \frac{\partial \phi}{\partial x} \\
\frac{d}{dt} H = \psi_0 \frac{\partial L}{\partial t} + \psi \cdot \frac{\partial \varphi}{\partial t} + \lambda \cdot \frac{\partial \phi}{\partial t} \, ,
\end{gathered}$$

and one concludes that (13) is equivalent to

$$\frac{d}{dt} \left(\psi(t) \cdot \left.\frac{\partial X}{\partial s}\right|_{s=0} - H \left.\frac{\partial T}{\partial s}\right|_{s=0} \right) = 0 \, .$$

The proof is complete. □

Remark 3.1. Theorem 3.1 is still valid in the situation where the boundary values of the state variables are not fixed. We have considered conditions (4) only to simplify the presentation of the maximum principle: transversality conditions are not relevant in the proof of our result.

Remark 3.2. It is possible to give a formulation of Theorem 3.1 under gauge-variance (see [5] for the concept of gauge-variance), and deal with equalities (6)–(8) up to first-order terms in the parameter s (see the quasi-invariance notion introduced in [9] for the unconstrained optimal control problem).

We now illustrate the application of Theorem 3.1 with an example.

4 Extraction of an Exhaustible Resource

We borrow from [2, pp. 194–198] a simple example of an optimal control problem in economics with one state variable, two control variables, and an equality constraint on the state and control variables ($n = 1$, $r = 2$, $m = 1$, $m' = 0$):

$$\begin{gathered}
\int_0^T u_1^\gamma(t) \mathrm{d}t \to \max \\
\dot{x}(t) = -u_2(t) \, , \\
x^{\alpha\gamma}(t) u_2^{\beta\gamma}(t) - u_1^\gamma(t) = 0 \, , \\
x(0) = x_0 \, , \quad x(T) = x_T \, ,
\end{gathered} \qquad (14)$$

where $\gamma < 1$, $\alpha + \beta < 1$ $(\alpha, \beta > 0)$, and $x_T < x_0$. Here $x(t)$ denote the stock of an exhaustible resource at time t; $u_2(\cdot)$ the rate of extraction from the stock; and $u_1(\cdot)$ the flow of consumption of the finished good: see [2] for more details on the model, and for an economic interpretation of the Pontryagin maximum principle. Problem (14) is invariant under the one-parameter family of transformations $h^s(t, x, u_1, u_2) = (T(t, s), X(x, s), U_1(u_1, s), U_2(u_2, s))$ defined by

$$\begin{aligned} T(t,s) &= \mathrm{e}^{-\gamma(\alpha+\beta)s} t \,, \quad X(x,s) = \mathrm{e}^{(1-\beta\gamma)s} x \,, \\ U_1(u_1,s) &= \mathrm{e}^{(\alpha+\beta)s} u_1 \,, \quad U_2(u_2,s) = \mathrm{e}^{(\alpha\gamma+1)s} u_2 \,, \end{aligned} \tag{15}$$

which coincides, for $s = 0$, with the identity transformation: $h^0(t, x, u_1, u_2) = (t, x, u_1, u_2)$. In fact, for problem (14) one has $L(u_1) = u_1^\gamma$, $\varphi(u_2) = -u_2$, and $\phi(x, u_1, u_2) = x^{\alpha\gamma} u_2^{\beta\gamma} - u_1^\gamma$, and all conditions (6), (7), and (8) are satisfied under (15):

$$\begin{aligned} L(U_1)\frac{d}{dt}T(t,s) &= \mathrm{e}^{\gamma(\alpha+\beta)s} u_1^\gamma \mathrm{e}^{-\gamma(\alpha+\beta)s} = u_1^\gamma \\ &= L(u_1) \,, \\ \varphi(U_2)\frac{d}{dt}T(t,s) &= -\mathrm{e}^{(\alpha\gamma+1)s} u_2 \mathrm{e}^{-\gamma(\alpha+\beta)s} = -\mathrm{e}^{(1-\beta\gamma)s} u_2 = \mathrm{e}^{(1-\beta\gamma)s}\dot{x} \\ &= \frac{d}{dt}X(x,s) \,, \\ \phi(X,U_1,U_2)\frac{d}{dt}T(t,s) &= \left(\mathrm{e}^{\alpha\gamma(1-\beta\gamma)s} x^{\alpha\gamma} \mathrm{e}^{\beta\gamma(\alpha\gamma+1)s} u_2^{\beta\gamma} - \mathrm{e}^{\gamma(\alpha+\beta)s} u_1^\gamma\right) \mathrm{e}^{-\gamma(\alpha+\beta)s} \\ &= \mathrm{e}^{\gamma(\alpha+\beta)s}\left(x^{\alpha\gamma} u_2^{\beta\gamma} - u_1^\gamma\right) \mathrm{e}^{-\gamma(\alpha+\beta)s} \\ &= \phi(x,u_1,u_2) \,. \end{aligned}$$

Having in mind that the problem is autonomous, that is, the Hamiltonian $H(t, x, u_1, u_2, \psi_0, \psi, \lambda) = \psi_0 u_1^\gamma - \psi u_2 + \lambda \left(x^{\alpha\gamma} u_2^{\beta\gamma} - u_1^\gamma\right)$ does not depend on t, and that from equality (5) this implies the Hamiltonian H to be constant along the extremals, it follows from our Theorem 3.1 that

$$(1 - \beta\gamma)\psi(t)x(t) + \gamma H(\alpha + \beta)t = constant \,. \tag{16}$$

Relation (16) gives an immediate insight about the solution of the problem – the rate of the value times the stock size of the resource must be constant: $\frac{d}{dt}\left(\psi(t)x(t)\right) = const.$ This conservation law has an economical interpretation in terms of the Cobb-Douglas form of the production function.

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